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THERMODYNAMIC PROPERTIES OF FLUID FLOW ACROSS A MAGNETIC FIELD

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(*Received August* 31, 1960)

ABSTRACT. This paper deals with the study of the various thermodynamic quantities like internal energy, enthalpy, entropy, etc., involved in the investigation of the flow of a conducting fluid in the presence of a uniform transverse magnetic field. The analogues of Rayleigh and Fanno lines readily follow from the basic equations. It is shown that the internal energy and enthalpy of an electrically conducting fluid obeying perfect gas depends, in the presence of a transverse magnetic field, on its density and the strength of the magnetic field. The entropy and the specific heat at constant volume do not seem to be affected by the presence of the magnetic field. The behaviour of the specific heat at constant pressure depends on which of the gas pressure and the total pressure is kept constant. A transverse magnetic field reduces the specific heat at constant gas pressure and the corresponding adiabatic constant by a factor proportional to the ratio of the magnetic pressure to the gas pressure. However, if the total pressure is kept constant, the magnetic field has no effect on the specific heat. Lastly, the effect of the magnetic field on the velocity of sound is discussed. In the limiting cases of weak and strong magnetic fields, the velocity of sound reduces to the ordinary sonic speed and the Alfvén speed respectively.

I. INTRODUCTION

The hydrodynamical motion of an electrically conducting fluid in the presence of a transverse magnetic field gives rise to induced electric currents which interact with the magnetic field to produce mechanical forces thereby affecting the fluid flow. Therefore, it is necessary to take account of this hydromagnetic interaction, and hence, the terms involving magnetic field appear in the equations governing the fluid flow (Alfvén, 1950; and Hoffmann and Teller, 1950). The investigation of the compressible fluid flow in the presence of a transverse magnetic field was initiated by Hoffmann and Teller (1950) who described the relativistic and nonrelativistic propagation of plane hydromagnetic shock waves in a medium of infinite electrical conductivity.

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In order to take account of the contribution of the magnetic field to the pressure and internal energy of the fluid, we define total pressure p^* and total internal enrgy E^* as

$$p^* = p + \frac{H^2}{8\pi} \qquad \ldots \quad (1)$$

and

$$E^* = E + \frac{H^2}{8\pi\rho} \qquad \ldots \quad (2)$$

The unstarred quantities p and E refer to the gas pressure and internal energy in the absence of the magnetic field H. These quantities are connected by the relation

$$E = \frac{1}{\gamma-1} \frac{p}{\rho} \qquad \ldots \quad (3)$$

In terms of the total pressure p , the total internal energy E^* is given by

$$E^* = \frac{1}{\gamma-1}\left(\frac{p^*}{\rho}\right) + \frac{\gamma-2}{\gamma-1}\left(\frac{H^2}{8\pi\rho}\right) \qquad \ldots \quad (4)$$

where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume.

Similarly, the total enthalpy h^* may be defined as

$$h^* = E^* + \frac{p^*}{\rho} = h + \frac{H^2}{4\pi\rho} \qquad \ldots \quad (5)$$

where h is the enthalpy in the absence of the magnetic field.
In terms of the Alfvén speed

$$V_{AU} = \left(\frac{H^2}{4\pi\rho}\right)^{\frac{1}{2}} \qquad \ldots \quad (6)$$

the expression for total enthalpy h^* becomes

$$h^* = h + V^2_{AU} \qquad \ldots \quad (7)$$

From the equations obtained for the limiting case of infinitesimal disturbances in the Hoffmann-Teller paper (1950), we may write the equations governing the thermodynamics of hydromagnetic fluid flow as the following:

The equation of continuity or constant mass flux

$$\rho v = \text{constant} = M \qquad \ldots \quad (8)$$

the equation of constant momentum flux

$$\rho v^2 + p^x = \bar{p}^x \tag{9}$$

and the equations

$$\tfrac{1}{2}v^2 + h^x = \bar{h}^x \quad \ldots \tag{10}$$

and

$$Hv = \text{constant} \quad \ldots \tag{11}$$

where $\bar{p}^x$ and $\bar{h}^x$ are the stagnation values of total pressure and total enthalpy. Eq. (10) is analogous to the first law of thermodynamics, while the Eq. (11) describes the relation of magnetic field strength and gas velocity v. From Eqs. (8) and (11) it follows that

$$\frac{H}{\rho} = \text{constant} = \alpha \tag{12}$$

In other words, the changes in magnetic field strength and density are such that the strength of the magnetic field is always proportional to density.

II. RAYLEIGH AND FANNO LINES

The elimination of v between Eqs. (8) and (9) yields

$$p^x + \frac{M^2}{\rho} = p + \frac{H^2}{8\pi} + \frac{M^2}{\rho} = \bar{p}^x \tag{13}$$

which is the analogue of the Rayleigh line in ordinary gasdynamics.

Similarly, the elimination of v between (8) and (10) gives

$$h^x + \frac{1}{2}\left(\frac{M}{\rho}\right)^2 = h + \frac{H^2}{4\pi\rho} + \frac{1}{2}\left(\frac{M}{\rho}\right)^2 = \bar{h}^x \quad \ldots \tag{14}$$

which is analogous to the customary Fanno line.

On expressing H in terms of ρ with the help of Eq. (12), these equations reduce to

$$p + \frac{\alpha^2}{8\pi}\rho^2 + \frac{M^2}{\rho} = \bar{p}^x \quad \ldots \tag{15}$$

and

$$h + \frac{\alpha^2}{4\pi}\rho + \frac{1}{2}\left(\frac{M}{\rho}\right)^2 = \bar{h}^x \quad \ldots \tag{16}$$

The constant α is determined from the initial value of the magnetic field and gas density when the gas is at rest.

III. CHANGE IN INTERNAL ENERGY

From Eq. (2) we have for the change in internal energy

$$dE^x = d\left(E+\frac{H^2}{8\pi\rho}\right)$$

which, with the help of Eq. (12), reduces to

$$dE^x = dE+\frac{H^2}{8\pi\rho^2}\,d\rho \qquad \dots \quad (17)$$

For a gas satisfying the perfect gas law

$$p = RT\rho \qquad \dots \quad (18)$$

with R and T as the universal gas constant and absolute temperature, the change in internal energy is given by

$$\left(\frac{\partial E^x}{\partial\rho}\right)_T = \left(\frac{\partial E}{\partial\rho}\right)_T+\frac{H^2}{8\pi\rho^2} = \frac{1}{2}\left(\frac{V^2_{Ay}}{\rho}\right) \qquad \dots \quad (19)$$

where V_{Ay} is the Alfvén speed. In the limit of zero magnetic field

$$\left(\frac{\partial E^x}{\partial\rho}\right)_T \rightarrow \left(\frac{\partial E}{\partial\rho}\right)_T = 0$$

Hence, the internal energy of a perfect gas of infinite electrical conductivity in the presence of a transverse magnetic field depends—unlike in the thermodynamics of ordinary fluid flow—on its density.

IV. CHANGE IN ENTHALPY

Similarly, on combining Eqs. (5) and (12), we have for the change in enthalpy

$$dh^x = \left(h+\frac{H^2}{4\pi\rho}\right)$$

$$dh+d\frac{H^2}{4\pi\rho^2}\,d\rho \qquad \dots \quad (20)$$

For a gas obeying Eq. (18), the last equation yields

$$\left(\frac{\partial h^x}{\partial \rho}\right)_T = \frac{H^2}{4\pi\rho^2} = \frac{V^2{}_{A\!f}}{\rho} \qquad \dots \quad (21)$$

and for $H \to 0$.

$$\left(\frac{\partial h^x}{\partial \rho}\right)_T \to \left(\frac{\partial h^*}{\partial \rho}\right)_T = 0$$

Therefore, it may be concluded that, like internal energy, the enthalpy also depends on the density of a conducting fluid in the presence of a transverse magnetic field.

V. CHANGE IN ENTROPY

Here it will be shown that the terms involving magnetic field do not appear in the expression for the change in entropy (Sen, 1956)

$$ds^x = \frac{dQ^x}{T} = \frac{1}{T}\left[dE^x + p^x d\left(\frac{1}{\rho}\right)\right] \qquad (22)$$

because, on substitution for p^x and dE^x from Eqs. (1) and (17), we have

$$ds^x = \frac{1}{T}\left[dE + \frac{H^2}{8\pi\rho^2}d\rho + pd\left(\frac{1}{\rho}\right) - \frac{H^2}{8\pi\rho^2}d\rho\right] \qquad (23)$$

$$= \frac{1}{T}\left[dE + pd\left(\frac{1}{\rho}\right)\right] = \frac{dQ}{T}$$

$$= ds$$

VI. SPECIFIC HEATS OF GAS

The specific heat of the fluid at constant volume is given by

$$C^x{}_{vol\cdot} = \left(\frac{\partial E^x}{\partial T}\right)_{vol\cdot} = \left(\frac{\partial E}{\partial T}\right)_{vol\cdot} = C_{vol\cdot} \qquad (24)$$

Therefore, the specific heat at constant volume is not affected by the magnetic field.

The influence of a transverse magnetic field on the specific heat at constant pressure depends, however, on which of the gas pressure p or the total pressure p^x is kept constant. If the gas pressure p is kept constant, then

$$C^x{}_p = \left(\frac{\partial h^x}{\partial T}\right)_{-} = C_p + \frac{H^2}{4\pi p^2}\left(\frac{\partial \rho}{\partial T}\right)_p \qquad \dots \quad (25)$$

which, for a perfect gas, becomes

$$C^x{}_p = C_p - 2R\left(\frac{H^2}{8\pi\rho}\right)$$

or $$C^x{}_n - C_n = -2R\,\frac{\text{Magnetic pressure}}{\text{Gas pressure}} \qquad \dots \quad (26)$$

Hence, the presence of a transverse uniform magnetic field reduces, in this case, the specific heat at constant gas pressure by an amount proportional to the ratio of the magnetic pressure and the gas pressure.

In the second case, when the total pressure p^x is kept constant, we similarly have

$$\begin{aligned} C^x{}_{p^x} &= \left(\frac{\partial h^x}{\partial T}\right)_p = \left[\frac{\partial}{\partial T}\left\{E + \frac{p^x}{\rho} + \frac{H^2}{8\pi\rho}\right\}\right]_{p^x} \\ &= C_{vol.} + p^x \frac{\partial}{\partial T}\left(\frac{1}{\rho}\right)_{p^x} + \frac{H^2}{8\pi\rho^2}\left(\frac{\delta\rho}{\partial T}\right) \qquad \dots \quad (27) \\ &= C_{vol.} + p\frac{\partial}{\partial T}\left(\frac{1}{\rho}\right) \end{aligned}$$

[by virtue of Eq. (12)]

$$= C_p$$

In this case, the terms involving magnetic field do not appear in the expression for the specific heat which has the same value as in the nonmagnetic case. Similarly, the adiabatic constant of the gas is influenced by the transverse magnetic field if the gas pressure is maintained constant.

VII. LOCAL VELOCITY OF SOUND

The new velocity of sound C^x is analogically defined with the help of Eqs. (1) and (12), as

$$C^x = (C^2 + V^2{}_{AV})^{\frac{1}{2}} \qquad (28)$$

where C is the velocity of sound in the absence of the magnetic field. In the case of magnetic field, the new velocity of sound in a conducting fluid is increased, the increment being proportional to the ratio of the magnetic pressure and the gas pressure. However, if the magnetic field exceeds the gas pressure considerably, the new velocity of sound approaches the Alfvén speed. The coincidence with the Alfvén speed is purely accidental. Although the sound waves appear to have precisely the same velocity as that of Alfvén waves, they are longitudinal in character whereas the Alfvén waves are transverse in nature. Similarly, the Mach number, in the case of very strong magnetic fields, is defined by the ratio of the flow speed and the Alfvén speed.

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